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The acoustic field of sources in shear flow with application to jet noise: convective amplification

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Lighthill, in his elegant and classic theory of jet noise, showed that the far-field acoustic pressure of noise generated by turbulence is proportional to the integral over the jet volume of the second time derivative of the Lighthill stress tensor, the integrand being evaluated at a retarded time. The purpose of this paper is to generalize the above results to include the effects of mean flow (velocity and temperature) surrounding the source of sound. It is shown quite generally that the integrand is now a certain functional of the Lighthill stress tensor evaluated at a retarded time. More important, however, at low and high frequencies this functional assumes an extremely simple form, so that the acoustic field can once more be given by integrals of the time derivatives of the Lighthill tensor. Both the self- and the shear-noise contributions to the pressure are evaluated.

1. Introduction

In a recent series of papers Mani (1972, 1974, 1975*a*, *b*) showed that, by including the effects of mean flow surrounding the source of sound, many seemingly strange characteristics of the noise of round jets could be explained in a systematic manner. Of course, Mani retained some of the original concepts of Lighthill (1952) (e.g. compact and convecting quadrupoles as sources of sound) but forcefully showed that acoustic/mean-flow interaction is a key physical phenomenon that cannot be ignored in jet noise, as is commonly done in applications of Lighthill's work.

The importance of acoustic/mean-flow interaction has been recognized for some time and views to that effect have been expressed by Ribner (1962), Csanady (1966), Pao (1973) and Gottlieb (1960). However, their discussions are focused on one limited aspect of mean-flow shrouding, namely shielding at high frequencies and shallow angles. More recently Ffowcs Williams (1974), starting from the exact Lighthill result, showed that a vortex-sheet analogy is naturally contained in the equations of motion. Again his work points to the importance of a shrouding mean flow.

Our primary interest here is in a discussion of the noise generated by *small-scale turbulence* surrounded by the mean flow. Thus we adopt the Lighthill picture of noise generation by reasonably small and incoherent eddies convecting with the fluid. The vortex-sheet analogy of Ffowcs Williams, on the other hand, suggests that some of the noise generated may indeed come from the "instability of the jet boundary". Here we shall say nothing about the latter source mechanism but recall that Mani's work (without any large-scale instability) is very successful in explaining most properties of the noise of round jets, hot or cold. Admittedly Mani's comparisons cover high subsonic to low supersonic jet velocities, and large-scale instability becomes most prominent at even higher jet velocities (Ffowcs Williams 1973).

Recognizing then that the following work may have Mach number limitations, we start from Lilley's (1972) equation, which takes the effects of mean flow into account to lowest order. More precisely, the Lilley equation is valid for unidirectional sheared flows with arbitrary velocity and temperature profiles. Solutions to this equation have been given by Mani in the special case of slug-flow profiles, by Goldstein (1975) for low frequencies and by the author (1976*a*) for high frequencies. The last two studies are for arbitrary velocity profiles and cold jets.

Each of the above authors has given the solution for a harmonically oscillating and convecting point singularity. The purpose of this paper is to obtain the *time-dependent* solution to Lilley's equation with the appropriate *shear-* and *self*-noise sources as forcing terms. It is shown that the acoustic pressure can be expressed as an integral over the jet volume of certain time derivatives of the Lighthill stress tensor evaluated at a retarded time. In this respect the present results are very reminiscent of those of Lighthill and indicate that the stable acoustic solutions to Lilley's equation probably describe jet noise due to small-scale convecting eddies.

2. Formulation and solution of the problem

As discussed in the previous section, our starting point is Lilley's equation

$$L(p; U, x') = \frac{1}{c^2} D_U^3 p - D_U \Delta p - \frac{d}{dr}(\log c^2) D_U \frac{\partial p}{\partial r} + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial x' \partial r} = \mathscr{S}(\mathbf{x}', t), \qquad (1a)$$

where p is the acoustic pressure, $U = U(r)$ and $c = c(r)$ are the undisturbed (i.e. mean) jet velocity and speed of sound respectively, Δ is the three-dimensional Laplacian in the space variables and

$$D_U = \partial/\partial t + U\partial/\partial x' \qquad (1b)$$

is a convective derivative. Physical space is spanned by a stationary cylindrical polar co-ordinate system $\mathbf{x}' = (r, \theta, x')$ and t denotes time (figure 1). The source strength $\mathscr{S}(\mathbf{x}', t)$ is given by

$$\mathscr{S} = \rho D_U \nabla' . \nabla' . \mathbf{u}'\mathbf{u}' - 2\rho \frac{dU}{dr}\frac{\partial}{\partial x'} \nabla' . u_r' \mathbf{u}', \qquad (2)$$

where $\rho = \rho(r)$ is the undisturbed density, ∇' is the gradient operator with respect to the $\mathbf{x}'$ co-ordinate system and $\mathbf{u}'$ is essentially the fluctuating turbulent velocity (u_r' is its radial component). Equations (1) and (2) describe *approximately*

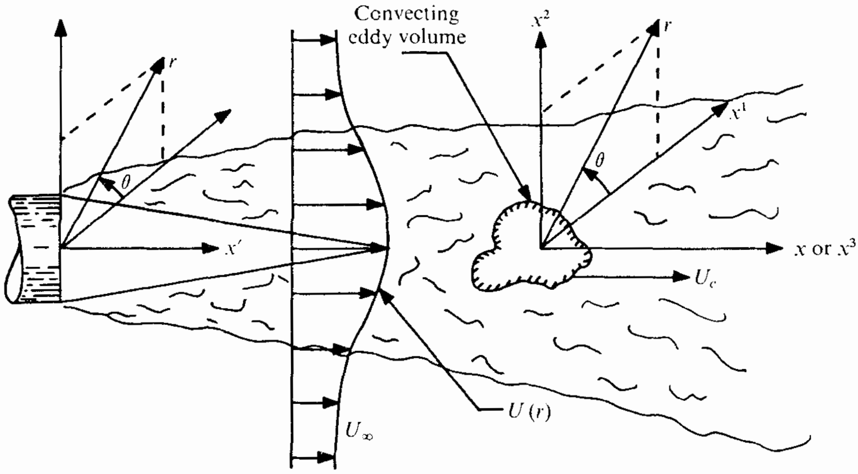

FIGURE 1. Geometry of the problem.

the propagation and generation of sound in a turbulent jet. The quantity $\mathbf{u'u'}$ is called the Lighthill stress tensor.

It should be pointed out that (1*a*) contains two fundamentally different types of quantity: deterministic variables (e.g. c and U) and random variables (e.g. p and $\mathscr{S}$). The purpose of this paper is to establish the dependence of the far-field instantaneous pressure on the instantaneous noise source. Once this dependence is known, the mean-square pressure (or suitable autocorrelations of the pressure) can be readily computed. This approach is exactly analogous to the one used by Ribner [1969, equations (1) and (3)].

Following the classical notions of Lighthill (1952), we assume that the source of sound is a convecting turbulent eddy whose velocity is $\mathbf{U}_c = (0, 0, U_c) =$ constant. Generally U_c is some fraction of the jet exit velocity (Davies, Fisher & Barratt 1963). Let us further assume that the 'spatial and temporal' characteristics of this eddy, when viewed from a reference frame $\mathbf{x} = (x^1, x^2, x^3)$ *attached* to the eddy, are given by

$$\mathbf{uu}(\mathbf{x}, t) = \mathbf{T}(\mathbf{x}, t), \tag{3a}$$

where $\mathbf{T}$ is a 'known' function. Typically $\mathbf{T}$ is given by results for isotropic turbulence (Proudman 1952; Ribner 1969). These remarks on source convection imply that the Lighthill stress tensor in Lilley's equation can be written as

$$\mathbf{u'u'} = \mathbf{T}(\mathbf{x'} - \mathbf{U}_c t, t). \tag{3b}$$

Of course, our remarks on representing a stochastic variable $\mathbf{uu}$ in terms of a 'known' function $\mathbf{T}$ are highly qualitative. It is well known that only the statistical properties of $\mathbf{uu}$ can be represented (with some degree of accuracy) by the results for isotropic turbulence. Nevertheless, it is possible to show that the conclusions we draw from this paper are exactly the same as those obtained by a more rigorous analysis involving moving correlations of the noise source term (see appendix).

It is convenient to introduce the Galilean transformation $\mathbf{x} = \mathbf{x'} - \mathbf{U}_c t$ into our governing equations. The final result is that the pressure fluctuations obey

$$L(p; V, x) = f(\mathbf{x}, t), \tag{4a}$$

where
$$f(\mathbf{x},t) = \underset{\text{self-noise}}{\rho D_V \nabla . \nabla . \mathbf{T}(\mathbf{x},t)} - \underset{\text{shear noise}}{2\rho \frac{dV}{dr}\frac{\partial}{\partial x}\nabla . \mathbf{T}_r(\mathbf{x},t)}, \tag{4b}$$
$$V = U - U_c, \tag{4c}$$

$\mathbf{T}_r = u_r \mathbf{u}$ is the radial component of $\mathbf{T}$ (u_r is the radial component of $\mathbf{u}$) and ∇ is the gradient operator with respect to the $\mathbf{x} = (x^1, x^2, x^3) = (r, \theta, x)$ co-ordinate system (figure 1). The space derivative in D_V is $\partial/\partial x$.

The first term on the right-hand side of (4*b*) is usually called the *self-noise source* and the second the *shear-noise source*. We shall retain this terminology even though it is somewhat misleading since, as we shall see, the self-noise terms also generate contributions to the acoustic pressure that are proportional to dV/dr. Note, however, that both source terms are *quadratic* in the velocity fluctuations. This definition of shear noise, essentially that given by Lilley (1972), differs from those given by Lighthill (1952) and Ribner (1969). The most significant difference is that our shear noise is quadratic in the velocity fluctuations, whereas the Lighthill–Ribner forms are only linear.

We next solve (4*a*) under the additional restrictions that p is finite on the jet axis $r = 0$ and represents outgoing waves at infinity. Furthermore, this solution will be valid only as $t \to \infty$ since all initial conditions (at $t = 0$) on p are ignored. In the following analysis we assume that $dU/dr < 0$, $dc/dr < 0$ and that $U = U_\infty$ = constant and $c = c_\infty$ = constant as $r \to \infty$. We also use the notation $M = U/c_\infty$ and $M_\infty = U_\infty/c_\infty$. These velocity and temperature profiles are representative of those of round jets in a large wind tunnel of speed U_∞ and temperature c_∞.

The solution to (4*a*) is obtained by a sequence of Fourier transformations. A similar approach was used by Pao (1973) for the Phillips equation. Define a three-dimensional transform of the pressure by

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,dt \int_{-\infty}^{\infty} e^{-isx}\,dx \int_{-\pi}^{\pi} e^{in\theta} p(r,\theta,x,t)\,d\theta, \tag{5}$$
$$-\infty < \omega < \infty, \quad -\infty < s < \infty, \quad n = 0, \pm 1, \pm 2, \ldots,$$

with an identical equation holding for F in terms of f, and apply this transform to (4*a*). After a number of integrations by parts (and ignoring the contributions from the limits $\pm\infty$) we find that

$$\frac{d}{dr}\left(v\frac{dP}{dr}\right) + v\left(k^2 g^2 - \frac{n^2}{r^2}\right)P = \frac{r(c/c_\infty)^2}{ic_\infty k(1-N\sigma)^3}F, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{6a}$$

where
$$v(r) = \frac{r(c/c_\infty)^2}{(1-N\sigma)^2}, \quad g^2 = \frac{(1-N\sigma)^2}{(c/c_\infty)^2} - \sigma^2, \tag{6b,c}$$
$$N = V/c_\infty = (U-U_c)/c_\infty, \tag{6d}$$
$$k = \omega/c_\infty \quad (-\infty < k < \infty), \quad \sigma = s/k \quad (-\infty < \sigma < \infty) \tag{6e,f}$$

and ω and s are the Fourier transform variables in (5). We shall refer to the Fourier time transform variable ω as the frequency.

Consider next two linearly independent solutions of the homogeneous version of (6*a*), and denote these by $\mathscr{J}_n(r)$ and $\mathscr{H}_n(r)$ respectively. We require these to have the limiting behaviour

$$\mathscr{J}_n(r) \sim r^{|n|} \quad \text{as} \quad r \to 0 \quad \text{(finiteness condition)}, \tag{7a}$$

$$\mathscr{H}_n(r) \to H_n^{(1)}(kg_\infty r) \quad \text{as} \quad r \to \infty \quad \text{(outgoing-wave condition)}, \tag{7b}$$

where $H_n^{(1)}$ is a Hankel function and g_∞ is the value of g at $r = \infty$. (Note that we restrict our attention to those values of σ for which g_∞^2 is positive; otherwise P is exponentially small in the far field). Using (6a) and (7) it is a classical Sturm–Liouville problem to show that (Friedman 1956, p. 151)

$$\text{Wronskian}\,(\mathscr{J}_n, \mathscr{H}_n) = \mathscr{J}_n(r)\frac{d\mathscr{H}_n(r)}{dr} - \frac{d\mathscr{J}_n(r)}{dr}\mathscr{H}_n(r) = \frac{w(k,\sigma,n)}{v(r)}, \tag{8a}$$

where w is *independent* of r and

$$P = \frac{\mathscr{H}_n(r)}{ic_\infty k\, w(k,\sigma,n)}\int_0^\infty \frac{r_0\,(c_0/c_\infty)^2}{(1-N_0\sigma)^3}\mathscr{J}_n(r_0)\,F_0\,dr_0, \tag{8b}$$

where the subscript zero denotes the value of the quantity at r_0. We remark that the upper limit of integration is really finite (since F vanishes outside the jet) and (8b) is valid as long as r is greater than this upper limit.

The result for the acoustic pressure p is readily obtained by applying the inverse transform corresponding to (5) to (8b). An intermediate result is

$$p = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega \int_{-\infty}^{\infty} e^{isx}\,ds \sum_{n=-\infty}^{\infty} e^{-in\theta}P. \tag{9a}$$

For large values of $(x^2+r^2)^{\frac{1}{2}}$, i.e. for a point in the far field, it is possible to evaluate the s integral in (9a) by the method of stationary phase (Carrier, Krook & Pearson 1966, p. 273). The point of stationary phase is given by

$$\sigma = s/k = \cos\Theta/(1-\tilde{M}_c\cos\Theta), \tag{9b}$$

where $\tilde{M}_c = (U_c - U_\infty)/c_\infty$ and Θ is the angle between the jet axis and the vector that joins the observation point and the origin of the convecting $\mathbf{x}$ co-ordinate system at the time of emission. In jet-noise theory Θ is usually interpreted as the angle with respect to the jet axis.

We remark that in the rest of this paper, unless otherwise noted, whenever σ (or $s = \sigma k$) appears it is to be replaced by the right-hand side of (9b).

After replacing F_0 in (8b) by the transform of f_0 [see (5)], substituting the resultant equation for P into (9a), carrying out the stationary-phase calculations and collecting terms, we arrive at

$$p = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} d\omega \int d\mathbf{x}_0\, G(r,\theta-\theta_0,x;r_0\,|\,t,\omega)\exp(-ik\sigma x_0) \times \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp(i\omega t_0) f(\mathbf{x}_0,t_0)\,dt_0, \tag{9c}$$

where $$G(r,\theta-\theta_0,x;r_0\,|\,t,\omega) = \frac{i\,e^{-i\omega t}}{4\pi c_\infty kR}\left(\frac{c_0}{c_\infty}\right)^2\frac{(1-\tilde{M}_c\cos\Theta)^2}{(1-\tilde{M}_0\cos\Theta)^3}e^{ikR} \times \sum_{n=-\infty}^{\infty}\exp[-in(\theta-\theta_0+\tfrac{1}{2}\pi)]\frac{\mathscr{J}_n(r_0)}{\pi w(k,\sigma,n)/2i}. \tag{9d}$$

where $d\mathbf{x}_0 = r_0\,dr_0\,d\theta_0\,dx_0$, the subscript zero on c and $\tilde{M}$ indicates their values at r_0, R is the distance between the point $\mathbf{x} = 0$ at the time of emission and the location of the observer (in the far field), $\tilde{M}_0 = M(r_0) - M_\infty$ ($M = U/c_\infty$, $M_\infty = U_\infty/c_\infty$) and $k = \omega/c_\infty$. The integral with no limits is to be taken over the jet volume. Note that the tilde denotes a non-dimensional velocity (normalized by c_∞) *relative* to $M_\infty = U_\infty/c_\infty$. Also, (9$c$, d) are to be evaluated at the point of stationary phase [i.e. σ is given by (9b)].

The function G is called the Green's function. Its importance is that once it is known the acoustic pressure of an arbitrary source $f(\mathbf{x}, t)$ can be calculated by quadrature from (9c). Replacing $f(\mathbf{x}, t)$ by $\delta(x)\,\delta(r-\bar{r})\,\delta(\theta)\exp(-i\Omega t)/r$ and evaluating the integrals in (9c) we find that $G(r, \theta, x; \bar{r}|t, \Omega)$ satisfies (4a) with its right-hand side replaced by the above combination of delta functions.

We emphasize that (9c, d) approximate the acoustic pressure only in the far field as $R \to \infty$.

We next write the expression for the acoustic pressure (9c) as a volume integral of a suitable term evaluated at the retarded time. At this point it is convenient to treat the self- and shear-noise terms separately [see (4b)]. First let us look at the self-noise. For brevity set

$$\nabla.\nabla.\mathbf{T}(\mathbf{x},t) = \mathscr{F}(\mathbf{x},t). \tag{10a}$$

From (4b), (9c, d) and (10a) it is possible to show that

$$p_{\text{self}} = \frac{1-\tilde{M}_c\cos\Theta}{4\pi R}\,\rho\left(\frac{c}{c_\infty}\right)^2 \int \frac{\hat{\mathscr{F}}[\mathbf{x}_0, t-R/c_\infty + \mathbf{x}_0.\boldsymbol{\zeta}/c_\infty(1-\tilde{M}_c\cos\Theta)]}{(1-\tilde{M}_0\cos\Theta)^2}\,d\mathbf{x}_0 \tag{10b}$$

by writing $\rho D_V\,\mathscr{F}(\mathbf{x},t)$ for $f(\mathbf{x}, t)$ and integrating (9c) by parts with respect to t_0 and x_0 to eliminate the operator D_V. Since in the Lilley formulation ρc^2 is a constant (proportional to the undisturbed static pressure) this quantity has been removed from under the integral sign. In (10b) we have also used the definition

$$\begin{aligned}\hat{\mathscr{F}}(\mathbf{x}_0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathscr{F}(\mathbf{x}_0,t_0)\,dt_0\int_{-\infty}^{\infty}\exp[i\omega(t_0-t)]\\ \times\exp\left[ik\frac{\Gamma}{1-\tilde{M}_c\cos\Theta}r_0\cos(\theta-\theta_0)\right]\sum_{n=-\infty}^{\infty}\exp[-in(\theta-\theta_0+\tfrac{1}{2}\pi)]\\ \times\frac{\mathscr{I}_n(r_0)}{\pi w(k,\sigma,n)/2i}\,d\omega.\end{aligned} \tag{10c}$$

Here Γ is an arbitrary quantity introduced for later convenience and

$$\boldsymbol{\zeta} = (\Gamma\cos\theta, \Gamma\sin\theta, \cos\Theta), \tag{10d}$$

where the components of $\boldsymbol{\zeta}$ are written in the order of (x^1, x^2, x^3); see figure 1. Note that to derive (10b) we have multiplied and divided G by

$$\exp\,[ikr_0\Gamma\cos(\theta-\theta_0)/(1-\tilde{M}_c\cos\Theta)].$$

The multiplicative factor is shown explicitly in (10c) and the divisor combines with the exp $(-ik\sigma x_0)$ in (9c) to form the retarded-time contribution $\mathbf{x}_0.\boldsymbol{\zeta}/[c_\infty(1-\tilde{M}_c\cos\Theta)]$.

By a completely similar procedure, we find that the shear-noise contribution is

$$p_{\text{shear}} = \frac{\cos\Theta(1-\tilde{M}_c\cos\Theta)}{2\pi R}\,\rho\left(\frac{c}{c_\infty}\right)^2\int\left(\frac{dN}{dr}\right)_0\frac{\hat{\mathscr{G}}[\mathbf{x}_0,t-R/c_\infty+\mathbf{x}_0.\boldsymbol{\zeta}/c_\infty(1-\tilde{M}_c\cos\Theta)]}{(1-\tilde{M}_0\cos\Theta)^3}\,d\mathbf{x}_0, \tag{11a}$$

where $\hat{\mathscr{G}}$ is given in terms of $\mathscr{G}$ by (10c) and

$$\mathscr{G}(\mathbf{x},t) = \nabla . \mathbf{T}_r(\mathbf{x},t). \tag{11b}$$

Equations (10b) and (11a) are the key results on which all the following developments of this paper are based. These results express the acoustic pressure in the far field ($R \to \infty$) as an integral of a suitable term over the jet volume. This term is to be evaluated at the retarded time, indicated explicitly in (10b) and (11a). In these respects, the present theory is extremely reminiscent of the classical work of Lighthill. There are, however, some important differences, which we shall point out in the following sections.

We remind the reader that in these equations Θ is the angle with respect to the jet axis and

$$M_\infty = U_\infty/c_\infty, \quad M_c = U_c/c_\infty, \quad \tilde{M}_c = M_c - M_\infty, \quad N = (U-U_c)/c_\infty, \tag{12a–d}$$

$$M(r) = U(r)/c_\infty, \quad \tilde{M}(r) = M(r) - M_\infty. \tag{12e,f}$$

In the following sections, we shall show that under suitable conditions the integrand in (10b) can be written as

$$\hat{\mathscr{F}}/(1-\tilde{M}\cos\Theta)^2 = H(r)\mathscr{F} = H(r)\nabla.\nabla.\mathbf{T} \tag{13}$$

for some function $H(r)$. $\mathbf{T}$ is the Lighthill stress tensor. Note that (13) is to be evaluated at a retarded time and that the double divergence operates only on the first argument of $\mathbf{T}$. In other words, $\mathbf{T}$ evaluated at the retarded time has the functional form $\mathbf{T} = \mathbf{T}[\mathbf{x},\ t - R/c_\infty + \mathbf{x}.\boldsymbol{\zeta}/c_\infty(1-\tilde{M}_c\cos\Theta)]$ but the operator $\nabla = \partial/\partial x_i$ in (13) operates only on the first argument of $\mathbf{T}$ (i.e. $\mathbf{x}$) and not on the second (i.e. t). Of course, the second argument of $\mathbf{T}$ is also a function of $\mathbf{x}$ because of retarded-time effects. The volume integral of (13) can be evaluated by two integrations by parts and two applications of the divergence theorem. Some of the volume integrals (after being converted to surface integrals) will vanish because the source function is zero far enough away from the jet. If we write (for two arbitrary functions ψ and ϕ) $\psi \overset{\int}{=} \phi$ whenever $\int \psi\, d\mathbf{x} = \int \phi\, d\mathbf{x}$ over a fixed volume, it is relatively straightforward to show that

$$H\nabla.\nabla.\mathbf{T} \overset{\int}{=} \mathbf{T}:\nabla\nabla H + \frac{2}{c_\infty(1-\tilde{M}_c\cos\Theta)}\nabla H.\mathbf{T}_t.\nabla\tau^* + \frac{H}{c_\infty^2(1-\tilde{M}_c\cos\Theta)^2}\nabla\tau^*.\mathbf{T}_{tt}.\nabla\tau^* + \frac{H\mathbf{T}_t:\nabla\nabla\tau^*}{c_\infty(1-\tilde{M}_c\cos\Theta)}, \tag{14}$$

where the subscript t denotes partial differentiation with respect to time and $\tau^* = \mathbf{x}.\boldsymbol{\zeta}$.

Similarly we shall show that the integrand in (11a) may be written as

$$\frac{(dN/dr)\mathscr{G}}{(1-\tilde{M}\cos\Theta)^3} = K(r)\mathscr{G} = K(r)\nabla.\mathbf{T}_r \tag{15}$$

for some function K, so that

$$K\nabla . \mathbf{T}_r \stackrel{\int}{=} -\nabla K . \mathbf{T}_r - \frac{K}{c_\infty(1-\tilde{M}_c\cos\Theta)}(\mathbf{T}_r)_t . \nabla\tau^*, \tag{16}$$

where $\mathbf{T}_r = u_r\,\mathbf{u}$ is the radial component of $\mathbf{T}$ and the subscript t again denotes differentiation with respect to time. Here u_r is the radial component of the velocity $\mathbf{u}$.

3. The Lighthill results

To see the significance of the previous results, let us consider them in a relatively simple and straightforward setting. We shall assume in this section that $M(r) \equiv 0$ ($M_c \neq 0$) and $c/c_\infty = 1$. Formally these simplifications correspond to those Lighthill invoked in his classic theory of jet noise. The function g in (6c) is now independent of r and Γ is chosen to be $\sin\Theta$. This yields

$$g = \sin\Theta/(1-M_c\cos\Theta), \quad \nabla\nabla\tau^* = 0. \tag{17a}$$

Here g has been evaluated at the point of stationary phase [see (9b)]. Two linearly independent solutions of (6a) satisfying (7a, b) are

$$\mathcal{J}_n(r) = J_n(kgr), \quad \mathcal{H}_n(r) = H_n^{(1)}(kgr), \tag{17b,c}$$

where J_n is a Bessel function of the first kind. Thus

$$w\pi/2i = (1-M_c\cos\Theta)^2 \tag{18a}$$

and the infinite series and integrals in (10c) can be evaluated in closed form to yield

$$\hat{\mathcal{F}}(\mathbf{x},t) = \mathcal{F}(\mathbf{x},t)/(1-M_c\cos\Theta)^2. \tag{18b}$$

After substituting (18b) into (10b) we find that

$$p_{\mathrm{self}} = \frac{\rho_\infty}{4\pi R(1-M_c\cos\Theta)}\int\mathcal{F}\left[\mathbf{x}_0, t-R/c_\infty+\frac{\mathbf{x}_0 . \boldsymbol{\zeta}}{c_\infty(1-M_c\cos\Theta)}\right]d\mathbf{x}_0. \tag{18c}$$

A comparison of the integrand in (18c) and (13) clearly shows that $H \equiv 1$, so that an alternative expression for the acoustic pressure is given by a combination of (14) and (18c):

$$p_{\mathrm{self}} = \frac{\rho_\infty}{4\pi R c_\infty^2(1-M_c\cos\Theta)^3}\boldsymbol{\zeta\zeta}:\int\mathbf{T}_{tt}\,d\mathbf{x}, \tag{19}$$

where the integrand is evaluated at the retarded time. Equation (19) is the very famous result of Lighthill. A thorough discussion and the implications of this expression were given by him over twenty years ago. Here, of course, $\boldsymbol{\zeta} = (\sin\Theta\cos\theta, \sin\Theta\sin\theta, \cos\Theta)$.

4. Low frequency results

In a recent paper, the author (1975) considered the low frequency radiation by a convecting source immersed in a jet with a constant (or slug) velocity profile.

Some of the general results of that paper are applicable to jets with *arbitrary*† velocity and temperature profiles. We shall invoke these results to establish the *lowest-order* acoustic field as $k \to 0$ for these more general profiles.

More precisely, our low frequency theory is an expansion in the small non-dimensional parameter $|ak|$, where a is a typical length scale associated with the velocity and temperature profiles (usually a can be taken to be the radius of the jet). The smallness of the above parameter implies that the wavelength of radiation is much larger than the radius of the jet. This expansion in $|ak|$ is singular in the sense of matched asymptotic expansions. The appropriate length scales in the inner and outer regions are the jet radius and the wavelength respectively. We refer the interested reader to the cited work for details.

First, it is known that to lowest order in $|ak|$ the radiation field will be axially symmetric, which implies that all terms except the $n = 0$ term can be ignored in (9d) and (10c). Furthermore, from (6a) and (7a) it may be seen that when $|k| \ll 1$

$$\mathscr{J}_0(r) = \text{constant} = 1 \tag{20a}$$

and

$$\mathscr{H}_0(r) = \frac{2i/\pi}{(1-N_\infty\sigma)^2}\int^r \frac{(1-N\sigma)^2}{r(c/c_\infty)^2}\,dr + \text{constant} \tag{20b}$$

$$= (2i/\pi)\log r + \ldots \quad \text{as} \quad r \to \infty. \tag{20c}$$

Now $\mathscr{H}_0(r)$ does not tend to $H_0^{(1)}(kg_\infty r)$ when $r \to \infty$, as required by (7b); rather it matches the Hankel function in the sense of matched asymptotic expansions. Thus, to lowest order in frequency, the correct inner solution *is* $\mathscr{H}_0(r)$ as given by (20b). Evaluation of the Wronskian (8a) shows that

$$\pi w/2i = (1-\tilde{M}_c\cos\Theta)^2, \tag{20d}$$

so that from (10c) we find that when $\Gamma = 0$

$$\hat{\mathscr{F}}(\mathbf{x},t) = \mathscr{F}(\mathbf{x},t)/(1-\tilde{M}_c\cos\Theta)^2. \tag{20e}$$

Second, at very low frequencies the *transverse* location (i.e. the x^1, x^2 coordinates) of the eddy volume is clearly unimportant (this is why the acoustic field of a source is nearly axially symmetric), so that Γ is indeed zero and

$$\boldsymbol{\zeta} \cong (0, 0, \cos\Theta). \tag{21}$$

We next use (20e) and (21) in (10b) [with a similar result for (11a)] and invoke (14) and (16) to arrive at

$$p = p_{\text{self}} + p_{\text{shear}} = \frac{\rho(c/c_\infty)^2\cos^2\Theta}{2\pi R c_\infty(1-\tilde{M}_c\cos\Theta)^2}\int (u_r u_x)_t \frac{dM/dr}{(1-\tilde{M}\cos\Theta)^3}\,d\mathbf{x}. \tag{22}$$

Here we write **uu** for the Lighthill stress tensor **T**, whose components in (22) are evaluated at the retarded time. The subscript t denotes partial differentiation with respect to time. We shall postpone the discussion of these results to a later section.

† I.e. continuously varying rather than slug profiles.

5. High frequency results

At the other end of the spectrum, at high frequencies, there is also a simple relationship between $\hat{\mathscr{F}}$ and $\mathscr{F}$. Here $|ka|$ is assumed to be large and the corresponding results are obtained from the lowest-order term in the expansion of the acoustic pressure in powers of $|ka|^{-\beta}$, $\beta \geqslant 0$.

Since $|ka|$ is large, the basic assumption of the theory is that the velocity and temperature profiles change only slightly in one wavelength. In order to derive an approximation to (6*a*) that is valid at high frequencies, it is necessary to compare the gradients of the mean-flow profiles directly with k. Note that (6*a*), as written, is unsuitable for this because dv/dr (i.e. the mean-flow gradient) multiplies dP/dr whereas k multiplies P. To place $v^{-1}dv/dr$ and k on an equal footing, introduce

$$\mathscr{P} = -r^{\frac{1}{2}}\frac{c}{c_\infty k}\frac{1}{1-N\sigma}P \tag{23a}$$

and substitute this into (6*a*) to obtain

$$\frac{d^2\mathscr{P}}{dr^2}+\left\{k^2g^2-\frac{n^2-\frac{1}{4}}{r^2}+\ldots\right\}\mathscr{P} = -\frac{c/c_\infty r^{\frac{1}{2}}}{ic_\infty k^2(1-N\sigma)^2}F, \tag{23b}$$

where the dots stand for particular combinations of the mean-flow gradients that are ignored in relation to k^2g^2 $(k\rightarrow\infty)$. Note that the first derivatives of $\mathscr{P}$ are absent from (23*b*).

It is easy to verify from the expressions (6*c*) and (9*b*) for g^2 and σ that the former becomes negative for values of Θ in the vicinity of the jet axis; in other words whenever Θ is in the zone of relative silence. In order to obtain the solution to (23*b*) which is uniformly valid for all r and Θ, we must consider r as a complex variable and pick suitable branches of the corresponding complex solution as g^2 changes sign. This can indeed be done in principle† (Carrier *et al.* 1966, p. 295; Balsa 1976*a*). However, in the present analysis we restrict our attention to $g^2 > 0$ to avoid this additional complication. This inequality means that we are looking at the sound pressure level and convective amplification outside the zone of relative silence. Now it is easy to see (and was actually proved by the author 1976*a*) that the same convective amplification factor appears inside the zone of silence as outside it. Thus the conclusions that we draw with respect to convective amplification are really valid for all angles. The *shielding* aspects of acoustic/mean-flow interaction, present only in the zone of relative silence, are discussed elsewhere (Balsa 1976*b*).

Since the solutions to the homogeneous version of (23*b*) are $(\xi/g)^{\frac{1}{2}}C_n(k\xi)$, where C_n is any Bessel function of the first kind and

$$\xi(r) = \int_0^r g\,dr, \tag{24}$$

† For example, the classical WKBJ turning-point conditions tell us how to go from one branch $(g^2>0)$ to the other $(g^2<0)$.

it is a simple matter to show from (23*a*) and (7*a*, *b*) that

$$\mathscr{J}_n(r) = \left(\frac{\xi}{gr}\right)^{\frac{1}{2}} \frac{c_J}{c} \frac{1-N\sigma}{1-N_J\sigma} J_n(k\xi), \tag{25a}$$

$$\mathscr{H}_n(r) = \left(\frac{\xi}{gr}\right)^{\frac{1}{2}} \frac{c_\infty}{c} \frac{1-N\sigma}{1+\tilde{M}_c\sigma} H_n^{(1)}(k\xi) \exp\left[-i\int_0^\infty (g-g_\infty)\,dr\right] \tag{25b}$$

and from (8*a*) that

$$\frac{w\pi}{2i} = \frac{c_J}{c_\infty} \frac{(1-\tilde{M}_c\cos\Theta)^2}{1-\tilde{M}_J\cos\Theta} \exp\left[-i\int_0^\infty (g-g_\infty)\,dr\right]. \tag{25c}$$

Here the subscript J denotes the value of a variable on the jet axis $r = 0$.

The infinite series and integrals in (10*c*) can again be evaluated in closed form. The result for the self-noise contribution is

$$p_{\text{self}} = \frac{\rho_\infty}{4\pi R(1-\tilde{M}_c\cos\Theta)} \int \left(\frac{c_\infty}{c_0}\right) \frac{(\xi_0/g_0 r_0)^{\frac{1}{2}}}{1-\tilde{M}_0\cos\Theta} \mathscr{F}[\mathbf{x}_0, t-R/c_\infty+\tau]\,d\mathbf{x}_0, \tag{26a}$$

where
$$\tau = \frac{\cos\Theta}{1-\tilde{M}_c\cos\Theta} \frac{x_0}{c_\infty} + \frac{\xi_0}{c_\infty}\cos(\theta-\theta_0) + \int_0^\infty (g-g_\infty)\frac{dr}{c_\infty}. \tag{26b}$$

Here the actual time delay τ differs somewhat from the classical result. This is because in the presence of mean-flow gradients the disturbance travels along the acoustic ray and *not* along a straight line joining the emission and observation points. Thus

$$\tau^* = x_0\cos\Theta + \xi_0(1-\tilde{M}_c\cos\Theta)\cos(\theta-\theta_0) + (1-\tilde{M}_c\cos\Theta)\int_0^\infty (g-g_\infty)\,dr, \tag{26c}$$

so that in this high frequency limit (retaining only the second time derivatives of the Lighthill tensor) we find from (26*a*) and (14) that

$$p_{\text{self}} = \frac{\rho_\infty}{4\pi R c_\infty^2(1-\tilde{M}_c\cos\Theta)^3} \int \frac{c_\infty}{c}\left(\frac{\xi}{gr}\right)^{\frac{1}{2}} \frac{d\mathbf{x}}{1-\tilde{M}\cos\Theta} \nabla\tau^*.(\mathbf{uu})_{tt}.\nabla\tau^*, \tag{27a}$$

where $\nabla\tau^*$ is the derivative of (26*c*) with respect to the source location. Here we write **uu** for the Lighthill stress tensor **T** and the subscript t again denotes partial differentiation with respect to time. The components of **uu** appearing in the various integrands are evaluated at the retarded time.

The shear-noise contribution is of lower order in k, so that the total acoustic pressure can be approximated by the self-noise component (i.e. $p \simeq p_{\text{self}}$).

When the sources are in the vicinity of the jet axis, (27*a*) assumes a simple form since $\xi_0 \simeq g_J r_0$, where g_J is the value of g on $r = 0$. In this case we have

$$p = \frac{\rho_\infty}{4\pi R c_\infty^2(1-\tilde{M}_c\cos\Theta)^3} \boldsymbol{\zeta}\boldsymbol{\zeta}: \int \frac{c_\infty}{c} \frac{d\mathbf{x}}{1-\tilde{M}\cos\Theta} (\mathbf{uu})_{tt}, \tag{27b}$$

where $\boldsymbol{\zeta} = [g_J(1-\tilde{M}_c\cos\Theta)\cos\theta,\quad g_J(1-\tilde{M}_c\cos\Theta)\sin\theta,\quad \cos\Theta]$.

Equation (27*b*) is extremely reminiscent of the Lighthill result and we propose its use at high frequencies. An equivalent form of (27*b*) was compared quite successfully with experimental data (Balsa 1976*a*).

6. Discussion

Let us first observe that, as we let $M \to 0$ and $c/c_\infty \to 1$ in the high frequency theory (27), we recover Lighthill's expression (19). At low frequencies [see (22)], however, we obtain $p = 0$. This is because the quadrupole source terms are $O(k^2)$ and terms of this order are ignored in the low frequency analysis.

At low frequencies the sound field p is of *dipole* type with an effective convective amplification factor of $(1 - \tilde{M}_c \cos \Theta)^{-5}$. Furthermore, this sound field is independent of jet temperature provided that the velocity fluctuations are nearly incompressible (i.e. that these fluctuations themselves do not depend on temperature). The last remark implies that the jet density exponent is zero. Thus, in the present formulation, at very low frequencies the mean-square acoustic pressure has an amplification factor of $(1 - \tilde{M}_c \cos \Theta)^{-9}$ [including the correction of $1 - \tilde{M}_c \cos \Theta$ for source volume effects (Ffowcs Williams 1963)]. Of course, the corresponding amplification factor for the mean-square pressure in the Lighthill theory is $(1 - \tilde{M}_c \cos \Theta)^{-5}$.

An equivalent form of (22) in frequency space for a cold jet was obtained by Goldstein (1975).

A very interesting result, alluded to previously, is that at low frequencies it is the self-noise contribution that generates (22) and the shear-noise term is completely cancelled by part of the self-noise term. In this sense, there is no shear noise at low frequencies in the *Lilley* formulation. It should be recalled, however, that the *Ribner* form of the shear noise survives at low frequencies (Ribner 1969). Consequently, when the effects of a shrouding mean flow are taken into account even the self-noise sources can generate terms proportional to the gradients of the mean flow. Mani (1975*b*) identified some of these additional terms, calculated them for slug flows and showed that they are needed to account for the negative density exponent of hot jets. The latter is observed experimentally (Hoch *et al.* 1973) and could not be explained theoretically before Mani's work.† However, Mani only accounted for the derivatives of the mean jet density. There are similar terms proportional to the velocity gradient. Also, as in the theory of Lighthill, it is precisely the variation of retarded time across the eddy volume that leads to a net acoustic field that behaves as R^{-1} $(R \to \infty)$ and to large convective amplification factors for higher-order singularities. The last remark also applies to the high frequency results.

In the purely formal theory of high frequency noise (i.e. $k \to \infty$, all other variables fixed) the radiation field is of quadrupole type with an effective convective amplification factor of $p \sim (1 - \tilde{M}_c \cos \Theta)^{-4}$. Thus one effect of the mean flow at high frequencies is to change the exponent of the convective amplification factor from -3 (Lighthill) to -4. This is exactly half-way between the classical and the previous low frequency results. The explicit dependence of the pressure on density is given by $p^2 \sim \rho$, which implies a jet density exponent of unity. Of course,

† It should always be remembered that the Lighthill equation (i.e. a rearrangement of the equations of motion) must be able to explain everything about jet noise. However, it may not be clear how to extract a certain piece of information from this equation whereas it may be clear how to extract it from another equation.

there is an additional (implicit) dependence on density through the quantity $\boldsymbol{\zeta}$. Observe, however, that the Θ dependence of the sound pressure field is changed somewhat from the Lighthill result because of the appearance of g_J (and hence Θ) in $\boldsymbol{\zeta}$.

It is now possible to use our low and high frequency expressions for the pressure to establish the dependence of the mean-square sound level on the turbulence properties. The analysis parallels the classical results of Ffowcs Williams (1963).

7. Conclusions

We have shown that at low and high frequencies the acoustic pressure in the far field can be written as an integral over the jet volume of the product of a suitable time derivative of the Lighthill stress tensor (evaluated at a retarded time) and a known function of r. The present high frequency results are valid outside the zone of silence. The most significant results are that a shrouding mean flow will change the exponent of the convective amplification factor from the classical results to some other value, that the self-noise source will generate terms proportional to the mean-flow gradients (hence the distinction between self- and shear-noise is artificial in the Lilley formulation) and that the convective amplification factor of a given kind of singularity is no longer a property of that singularity alone. For example, both a dipole and quadrupole may have an effective amplification exponent of -5 [see (22)]. Finally convective amplification itself is frequency dependent, having an index for the pressure of -5 and -4 at low and high frequencies respectively.

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Appendix†

The purpose of this appendix is to show the general mathematical relationships between the mean-square and instantaneous sound pressures.

Consider a linear wave operator $\mathscr{L}$ operating on the instantaneous pressure $p(\mathbf{x}, t)$: symbolically

$$\mathscr{L}[\partial/\partial t, \partial/\partial \mathbf{x}; \mathbf{a}]p = f(\mathbf{x}, t) = g(\mathbf{y} - \mathbf{U}_c t, \mathbf{Y}, t). \tag{A 1}$$

Here our physical space is spanned by a co-ordinate system $\mathbf{x}$ and t denotes time. The coefficients $\mathbf{a}$ of our wave operator are independent of time and of co-ordinates $\mathbf{y}$ that span a suitable subspace of $\mathbf{x}$. In other words, we write $\mathbf{x} = \mathbf{y} \cup \mathbf{Y}$, where $\mathbf{y}$ and $\mathbf{Y}$ are linearly independent subspaces of $\mathbf{x}$ such that the coefficients of $\mathscr{L}$ depend only on $\mathbf{Y}$, i.e. $\mathbf{a} = \mathbf{a}(\mathbf{Y})$. For example, in the Lilley equation $\mathbf{y}$ and $\mathbf{Y}$ may be identified with the axial and transverse co-ordinates respectively.

The right-hand side of (A 1) is a known source term that is assumed to convect with velocity $\mathbf{U}_c$ = constant in $\mathbf{y}$ space. Again, in the Lilley formulation, the

† The notation in this appendix is self-contained.

sources are generally convecting parallel to the axis of the jet.

If the Fourier time transform of p is denoted by p^*, then

$$p^*(\mathbf{x}|\omega) = \int d\mathbf{x}_0\, G(\mathbf{x},\mathbf{x}_0|\omega)\, g^*[\mathbf{y}_0, \mathbf{Y}_0\,|\omega(1-M_c\cos\Theta)], \tag{A 2}$$

where

$$\mathbf{x}_0 = \mathbf{y}_0 \cup \mathbf{Y}_0,$$

$$g^*(\mathbf{y},\mathbf{Y}|\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} g(\mathbf{y},\mathbf{Y},t)\, e^{i\omega t}\, dt, \tag{A 3}$$

$$p^*(\mathbf{x}|\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} p(\mathbf{x},t)\, e^{i\omega t}\, dt, \tag{A 4}$$

$M_c = U_c/c_\infty$ (c_∞ = speed of sound at infinity) and Θ is the angle between the observation and $\mathbf{U}_c$ directions. G is the Green's function that satisfies

$$\mathscr{L}[-i\omega, \partial/\partial\mathbf{x}; \mathbf{a}]\ G(\mathbf{x},\mathbf{x}_0|\omega) = \delta(\mathbf{x}-\mathbf{x}_0) \tag{A 5}$$

and all radiation and finiteness conditions. $\delta(\cdot)$ denotes a suitable multi-dimensional delta function and $\mathbf{x}_0$ is the location of the source.

Result (A 2) can be obtained by applying Fourier transforms to (A 1) in t, $\mathbf{y}$ space and evaluating the inversion integral by the method of stationary phase. As such, (A 2) is valid only in the far field as $x \to \infty$. It is also assumed that, as $x \to \infty$, $\mathscr{L}$ reduces to the classical wave operator.

Equation (A 2) shows that the far-field acoustic pressure (in frequency space) is given by the integral of the product of the Green's function and the spectrum of the source term in its moving reference frame. The integral is taken over the jet volume. Furthermore, it is the Doppler-shifted frequency $\omega(1-M_c\cos\Theta)$ of the integrand that contributes to the spectrum of the sound at frequency ω.

On the other hand, from the time-dependent solution $p(\mathbf{x},t)$ of (A 1) it is possible to show that

$$\Gamma^*(\mathbf{x}|\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int d\mathbf{x}_0\, |G(\mathbf{x},\mathbf{x}_0|\omega)|^2 H[\mathbf{x}_0, \omega\,\partial\Phi/\partial\mathbf{x}_0,\ \omega(1-\mathbf{U}_c\cdot\partial\Phi/\partial\mathbf{x}_0)], \tag{A 6}$$

where Γ^* is the Fourier transform (with respect to τ) of the correlation

$$\Gamma(\mathbf{x},\tau) = \int_{-\infty}^{\infty} p(\mathbf{x},t)\, p(\mathbf{x},t+\tau)\, dt. \tag{A 7}$$

As such, Γ^* is the spectrum of the mean-square acoustic pressure. The integral in (A 6) is taken over the jet volume. The definition of H involves the following string of quantities:

$$\mathscr{R}(\mathbf{y},\boldsymbol{\xi},\tau) = \int_{-\infty}^{\infty} f(\mathbf{y}+\tfrac{1}{2}\boldsymbol{\xi}, t+\tau)\, f(\mathbf{y}-\tfrac{1}{2}\boldsymbol{\xi}, t)\, dt, \tag{A 8}$$

$$R(\mathbf{y},\boldsymbol{\xi},\tau) = \mathscr{R}(\mathbf{y},\boldsymbol{\xi}+\mathbf{U}_c\tau,\tau) \tag{A 9}$$

and

$$H(\mathbf{y},\mathbf{k},\omega) = \int R(\mathbf{y},\boldsymbol{\xi},\tau)\, e^{-i\omega\tau}\, d\tau \int e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}. \tag{A 10}$$

Thus $\mathscr{R}$ is a suitable two-point correlation of the source term f with arbitrary time delay and spatial separation, R is the corresponding moving correlation and H is the frequency–wavenumber representation of this moving correlation.

The phase of the Green's function is given by Φ: specifically, we write

$$G(\mathbf{x}, \mathbf{x}_0|\omega) = |G(\mathbf{x}, \mathbf{x}_0|\omega)| \exp[-i\omega\Phi(\mathbf{x}, \mathbf{x}_0)].$$

One important assumption invoked in deriving (A 6) is that the amplitude of the Green's function changes negligibly in one correlation length of the source. We remark that (A 6) is valid both in the far and the near field. For a compact turbulent eddy, the second argument of H in (A 6) can be replaced by zero.

Equation (A 6) shows that the spectrum of the mean acoustic pressure can be obtained from the product of the Green's function and the Doppler-shifted turbulence spectrum. The resemblance between (A 2) and (A 6) is clear, so that it is indeed possible to draw qualitative conclusions from (A 2) for Γ^*. In particular, the expression for the far-field spectrum of a convecting source [see (A 2)] is qualitatively very similar to that of noise generated by turbulence [see (A 6)].

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